LONGITUDINAL DISPERSION IN ICE COVERED RIVERS

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ABSTRACT

Longitudinal dispersion is the spreading of suspended or dissolved substances caused by the combined action of differential advection and lateral mixing. Tracer tests in many rivers under open-water conditions have indicated that, initially, the temporal spread increases linearly with downstream distance (linear range) but settles down to a growth proportional to the square root of distance (Fickian range) very far away from the tracer injection point. Test data have been obtained in ice-covered rivers, starting in 1975. They invariably showed linear-range behaviour even though the test reaches were as long as hundreds of kilometres. As in the case of open-water conditions, the rates of spread under an ice cover are related to the friction factor of the flow, being somewhat higher than the open-water values for equal friction factors. To obtain data that extend into the Fickian range, it is suggested that tests be carried out in small streams, 10 to 20 metres wide. Based on early results in pipe flow, many water quality models assume Fickian dispersion but the field data suggest that this condition is not likely to be encountered in any but very small rivers. A more correct quantification of the dispersion process is outlined and its implications to the method of computation are discussed.

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INTRODUCTION

Water quality assessment and modelling in natural streams is often based on the assumption of one-dimensionality (eg. see Ambrose et al., 1986; Bowie et al., 1985; Schnoor et al., 1987). The three-dimensional equation expressing the mass conservation for any one water quality parameter is integrated over the flow depth and across the river width to arrive at a much simpler equation describing the cross-sectionally averaged concentration, C:

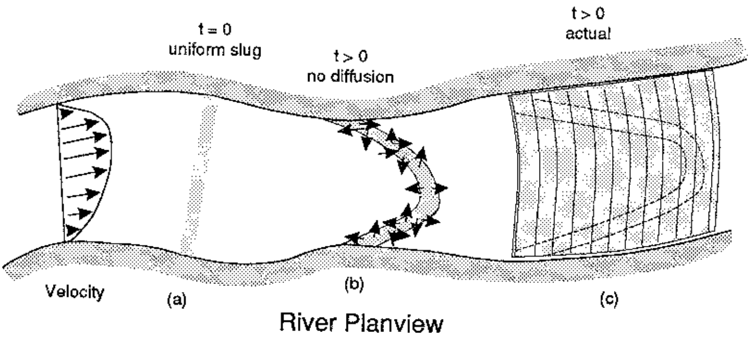
$$\partial(AC)/\partial t + \partial(AVC)/\partial x = \partial\{-A<(u-V)(c-C)>\}/\partial x + \Sigma S \qquad (1)$$

in which A=flow cross-sectional area; x, t=distance along the river and time; < > denotes spatial averaging over the cross-section of the flow; u=local value of streamwise velocity; V=<u>=average flow velocity in a given cross section; c=concentration at any one point in the flow; and ΣS represents the sum of the effects of various sources and sinks within the flow and along the flow boundaries, such as the river bed and the water surface. It is assumed that good mixing has been attained so that C is not too different from the concentration, c, prevailing at any point within the same cross-section.

In rivers, the velocity u varies vertically and laterally, causing differential advection of a contaminant which, in turn, results in longitudinal spreading that is far more pronounced than that due to turbulence. This effect is called longitudinal dispersion and is quantified by the first term on the right-hand-side of Eq. 1.

Figure 1 illustrates the natural mechanisms responsible for longitudinal dispersion. For simplicity, let it be assumed that a laterally uniform slug of a substance has been injected in the river at time zero, as depicted in sketch (a). If advection were the only process operating on the slug, the non-uniform river velocities would quickly distort it to the shape shown in sketch (b). However, molecular and turbulent diffusion act to reduce concentration gradients by transferring matter between regions of high and low concentrations, as suggested by the arrows in sketch (b). The combined effects of differential advection and diffusion, eventually result in a cloud that is almost uniformly spread across the river, and has a much greater longitudinal extent than would be the case if either advection or diffusion were to operate alone (sketch (c) in Fig. 1). Mass conservation dictates that the opposite trend applies to concentrations. Longitudinal dispersion is obviously an important diluting mechanism where slugs are injected in rivers such as accidental spills of toxic chemicals. Its importance diminishes as the time of release increases and becomes negligible for steady continuous release.

Fig. 1. Schematic illustration of longitudinal dispersion.



Taylor (1954) investigated longitudinal dispersion in pipe flows and proposed that, after a sufficient time has elapsed since the introduction of a substance in the flow, the dispersion term varies in proportion to $\partial C/\partial x$, x being the distance along the river. The coefficient of proportionality, D, was called the dispersion coefficient and postulated to be *independent of time and distance. Because of the similarity between* Taylor's concept and Fick's laws of diffusion, characterized by constant diffusion coefficients, the former is also called Fickian dispersion.

Taylor's theory was widely applied to rivers but field data gradually showed that riverine dispersion is seldom of the Fickian type. In many rivers, if it occurs at all, Fickian dispersion requires hundreds or even thousands of kilometres from the outfall to be established (Nordin and Sabol, 1974; Beltaos, 1980).

We already have a fair idea as to how to predict longitudinal dispersion in rivers under open water conditions (eg. see Elhadi et al, 1984) but, until recently, there has been almost a complete lack of field test data under an ice cover. Dispersion data in ice covered rivers are reviewed in this paper and research needs are assessed. The main implications of the existing data to numerical modelling of water quality in natural streams are explored.

BACKGROUND INFORMATION

The simplest situation for studying longitudinal dispersion is to consider the transport and spread of a substance which is introduced in a river as a slug, that is, a certain mass, M, is released more-or-less instantaneously. More complex releases can always be treated as a succession of elementary slugs and the results superimposed linearly.

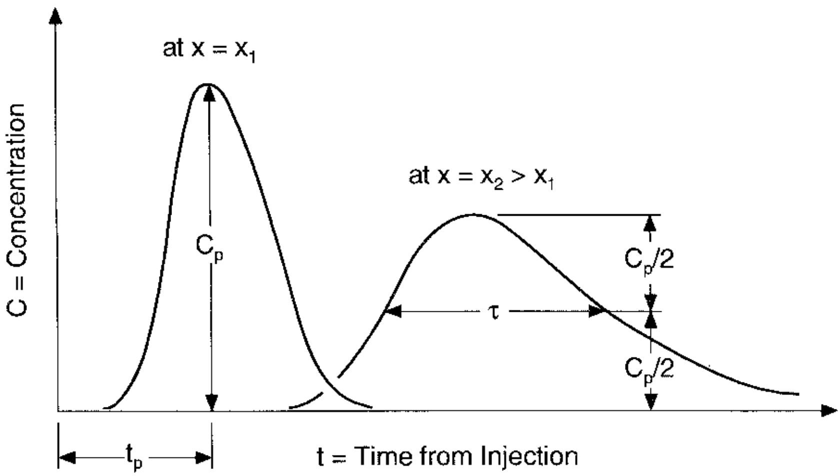
Some time after a slug release, the substance spreads all across the river and the average concentration C varies with time and distance as illustrated in Fig. 2. The function C(x,t) is adequately defined by the variation of the "half-spread", τ, the peak concentration, C_p, and the time to peak concentration, t_p, with the distance from the point of release, x. Once these quantities are known at any one value of x, the entire C-t curve can be obtained as explained in Beltaos (1980). In the same reference, t_p, C_p, and τ have been determined as follows.

$$t_p = x/V \tag{2}$$

$$\tau^2 = 11.10\ \beta\ (L/V)^2\{(x/L) + \exp(-x/L) + 1\} \tag{3}$$

$$C_p = 0.94\ M/Q\tau \tag{4}$$

Fig. 2. Schematic illustration of concentrations resulting from the release of a slug in a river.



In Eqs. 2-4, ß=dimensionless coefficient; L=a characteristic river length; V=average flow velocity; M=mass of contaminant in the slug; and Q=flow discharge. Equation 3 shows that, far from the source ($x \geq 3L$), the spread τ varies approximately as the square root of x which is characteristic of Fickian dispersion. The dispersion coefficient, D, can be calculated as

$$D = \beta LV \tag{5}$$

On the other hand, for x≤L, τ is roughly proportional to x, ie

$$\tau \approx \sqrt{5.55\beta}\ (x/V) = \sqrt{5.55\beta}t_p \tag{6}$$

and this range (0≤x≤L) has been termed the linear range. Analysis of numerous field tests performed under open-water conditions indicated that very seldom did the study reach extend far enough for the process to become Fickian. Often, it was only the linear range that could be tested (Beltaos, 1980). Essentially, this was due to the length, L, being much greater in rivers than what would have been expected for prismatic channels, as explained later.

The parameters ß and L depend on the hydraulic characteristics of the river. Theoretical considerations (Beltaos, 1980) have indicated that the former should increase with the velocity variance, ie

$$\beta = b <(u-V)^2>/V^2 \tag{7}$$

in which u=velocity at any one point in the flow; and b=dimensionless coefficient. Limited field data under open water conditions (Fig. 3) indicated that b = 0.025-0.1. However, the velocity variance requires elaborate measurements that are not usually performed. Alternatively, the variance is known to be approximately related to the friction factor of the flow, f (≡ $8V_*^2/V^2$; V_* =shear velocity), a quantity that is much easier to determine in practice. Consequently, ß has been plotted against V_*/V for over fifty open-water tests and, with three exceptions, the data points were within the band depicted in Fig. 4.

Beltaos (1980) showed that the length, L, is proportional to the travel distance required before the dispersing particles begin to move at the average stream velocity, V. In a prismatic channel, this would happen once a particle has been in transit long

Fig. 3. Variation of coefficient β with velocity variance. Hollow symbols denote open-water test results.

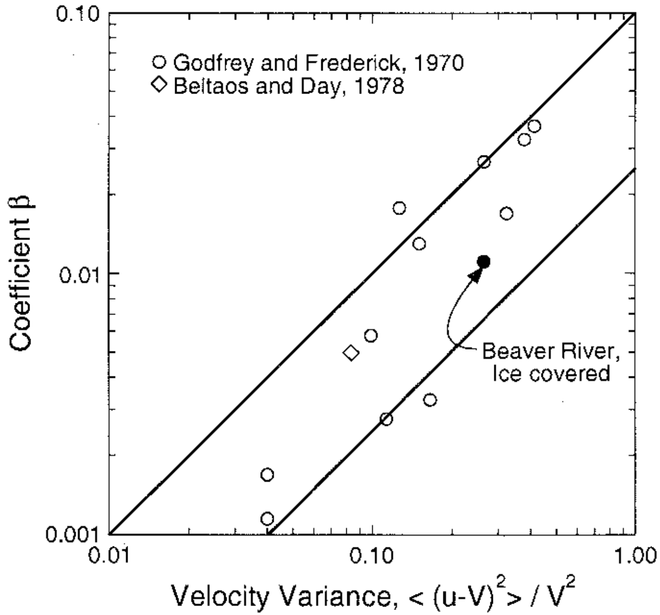
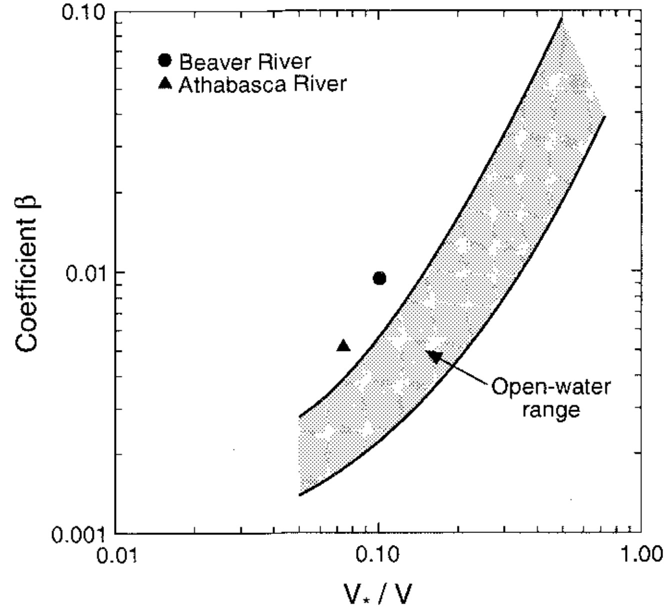


Fig. 4 Variation of coefficient β with V_*/V Solid symbols denote ice-covered flow conditions.



enough to diffuse laterally from one bank to the other, a few times over. In a natural stream, this is not sufficient because there are considerable variations in geometry and velocity along the river. Hence, L would have to be much greater than what would apply to the "equivalent" prismatic channel, that is, a prismatic channel having the reach-averaged hydraulics of the natural stream. With this background, it can be deduced that

$$L/W = a(W/H)(V/V_*) \tag{8}$$

and, using Eq. 5:

$$D/V_*H = d(W/H)^2 \tag{9}$$

in which W=channel width; H=average depth: and the coefficients a and d have been evaluated from the open-water test data to be mostly in the ranges 0.5-1.8 and 0.1-1.0 respectively. These values are much higher than those applicable to prismatic channels, i.e. $a \approx 0.15$ and $d \approx 0.1$. The scatter associated with the field values of b, a, and d reflects the fact that additional, so far unquantified, factors are at work, such as channel morphology, transverse mixing capacity, or presence of dead zones.

As an example of applying the above results, consider a large river, such that W=300 m and H=5 m. With typical values of a=1.0 and V/V_*=10, we calculate L=180 km, so that Fickian dispersion is not likely to be encountered before a distance of 540 km (!) from the point of release. Even in a small stream, with W=20 m and H=1 m, L amounts to 4 km and attainment of the Fickian range will require 12 km. These figures illustrate some of the limitations associated with the use of the conventional dispersion concept in water quality modelling.

DATA IN ICE COVERED RIVERS

The presence of an ice cover on a river changes the hydraulic conditions by imposing a second, occasionally very rough, boundary. This may influence the dispersion process but pertinent field testing did not commence until the mid-seventies. The available test results are reviewed in this section. All tests were carried out by injecting slugs of Rhodamine WT fluorescent dye, a relatively neutral and conservative tracer.

Beaver River near Beaver Crossing

In February 1975, a test was conducted by the writer on the Beaver River, Alberta, as part of a study that also included transverse mixing tests under ice-covered and open-water conditions (see also Beltaos, 1978). This reach comprises a single channel with regular meanders and has a sinuosity of 1.3.

The data indicated linear dispersion behaviour so that it was not possible to determine the length L; it can only be stated that L≥5.1 km, translating to a≥0.2 (see Eq. 8) which is compatible with the open-water range of a (0.5-1.8). From a plot of τ versus t_p (Fig. 5), it was found that ß=0.0095. Cross-sectional measurements of depth and velocity indicated that W=39.8 m, H=0.61 m, and velocity variance=0.27. The plot of t_p versus x in Fig. 6 indicated that V=0.24 m/s (see Eq. 2), while the river slope was taken as 0.21 m/km (Kellerhals et al., 1972). Consequently, $V_*=\sqrt{9.8x(0.61/2)0.00021} = 0.025$ m/s and $V_*/V=0.10$.

With this information, it can be further determined that the coefficient b (Eq. 7) is equal to 0.035 which is within the range of open-water test results (b=0.025-0.10; see also Fig. 3). However, when ß is plotted in Fig. 3, against the more readily obtainable parameter V_*/V, the data point falls above the open-water band.

Athabasca River from Bitumount to Embarras

Another test was conducted in February of 1978 downstream of Fort Mackay. Here again, it was only possible to sample within the linear range, even though the length of the test reach was 108 km (Beltaos, 1979). In this reach, the Athabasca River is relatively straight and flat, with numerous small and large islands.

Fig. 5. Increase of half-duration, τ, with time to peak concentration, Beaver River test.

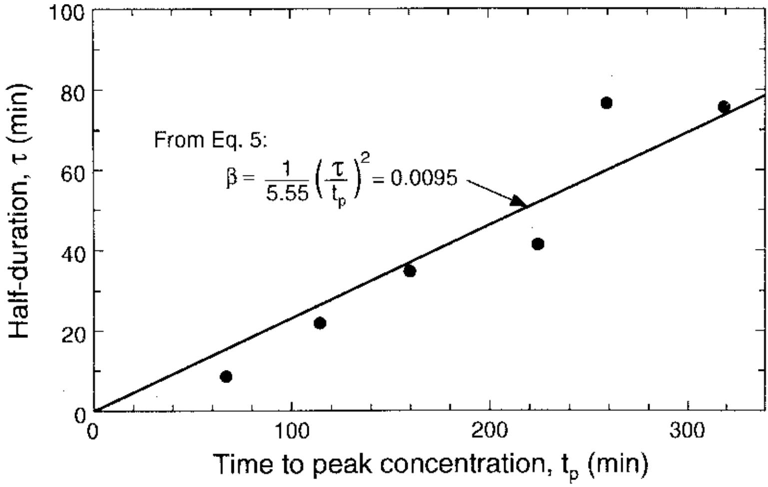
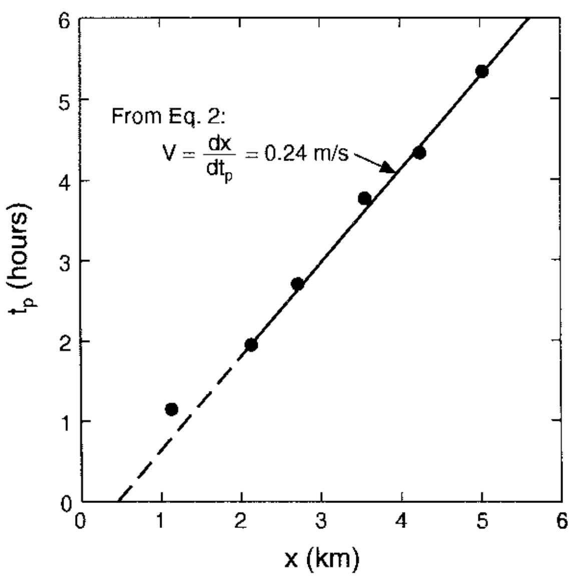


Fig. 6. Increase of time to peak concentration with distance, Beaver River test.



The salient hydraulic parameters were W=370 m, H=1.55 m, V=0.42 m/s, and V_*/V=0.071. The coefficient ß was evaluated as ß=0.0052 and this, too, plots relatively high in Fig. 4. It can also be calculated that the coefficient a would have to be on excess of 0.09.

Athabasca, Smoky, and Wapiti Rivers

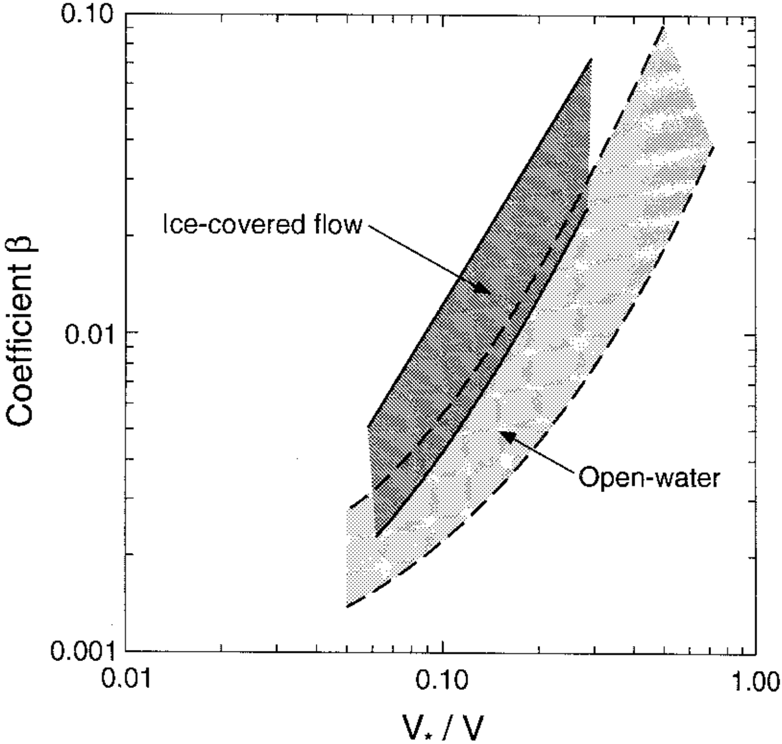
During 1989-1992 numerous tests were conducted in selected reaches of these rivers by Alberta Research Council (Van Der Vinne and Andres, 1990; Van Der Vinne, 1990; Van Der Vinne and Andres, 1992). Consecutive reaches of the Athabasca were tested, starting at Hinton and ending at Bitumount. The Wapiti test covered the reach between Grande Prairie and a little past the river mouth into the Smoky. Two additional tests covered the Smoky River from near the confluence of the Wapiti to the Smoky mouth on the Peace.

In these tests, river width ranged from 60 to 340 m while the average depth was between 0.6 and 1.4 m. Average velocities varied within the relatively narrow range of 0.3 to 0.5 m/s.

In all cases, the data exhibited linear behaviour, even though some of the test reaches were hundreds of kilometres long. With reference to Eq. 8, it is possible to calculate a value that the coefficient a would have had to exceed in each test. In general, these lower-bound values are less than 1.4 which does not contradict the open-water results (a≈0.5-1.8). However, there were two tests in which the coefficient a would have exceeded 3.2 and 2.8 respectively. These tests had the highest friction factors (f≈0.48 for both; Manning n≈0.07), owing to thick frazil accumulations under the ice cover. As discussed earlier, this would enhance the channel irregularity and thence the travel distance required for dispersing particles to sample a representative stream volume.

Using the data presented in the above noted references, the ß coefficients were plotted against V_*/V, thus defining the data band shown in Fig. 7. The open-water range is also plotted for comparison.

Fig. 7. Data range for ice-covered flow; coefficient β.



DISCUSSION

Figure 7 shows that, despite considerable overlap, the parameter ß is generally greater under an ice cover than in open water flow, for equal values of V_*/V, or of friction factors, f ($f \equiv 8V_*^2/V^2$). This is in accord with what would be expected from Eq. 7. It can be shown that, for the same value of the friction factor, the velocity variance would be generally greater under an ice cover than under open-water conditions. This is caused by differences between the friction factors f_i and f_b, and thence between the shear stresses, respectively associated with the ice and the river bed.

A rather striking result of the ice-covered flow tests is that they all exhibited linear dispersion behaviour. As mentioned earlier, the results of two of the tests hinted to larger values of L than in open-water flow. However, test data that will actually extend into the Fickian range are needed before definite conclusions can be drawn. The same applies to the conventional dispersion coefficient, D. It would be impractical to attempt such tests in large or even moderately sized rivers because of the large amounts of tracer and long sampling durations that would be required. Streams ranging in width between 10 and 20 m would be ideal, provided the cross-section and slope do not vary excessively. An initial estimate of L could be made using the open-water results and the test reach length set at five to ten times this estimate.

PEAK CONCENTRATIONS AND DYE QUANTITIES

In addition to the half-spread, τ, the peak concentration, C_p, is needed for dispersion predictions. Equation 4 shows that C_p varies in direct proportion to the mass of the injected slug and in inverse proportion to τ. The dispersing substance is assumed to be neutral, so that the possible losses due to various chemical or biological processes are not taken into account. In practice, however, losses are likely to occur, even with a relatively conservative tracer such as the Rhodamine WT dye used in the experiments. Therefore, it may be difficult to directly test the validity of Eq. 4.

Where losses are negligible or they are accounted for empirically, Eq. 4 has been found to perform adequately. For instance, the Beaver River test results indicated that $C_pQ\tau/M_r$ was 0.86, 0.88, 0.83, 0.68, 0.89, and 0.87 at the six sampling sites, close to the theoretical value of 0.94 (note that M_r denotes the recovered tracer mass). For the 1978 Athabasca River test, the three sampling sites indicated values of 0.84, 0.89, and 0.81.

A related practical question is how to determine the amount of dye that will be needed for a particular test, in order to produce measurable concentrations at the farthest downstream sampling site. For open-water conditions, Kilpatrick (1970) analyzed a large number of data and proposed the following empirical relationship:

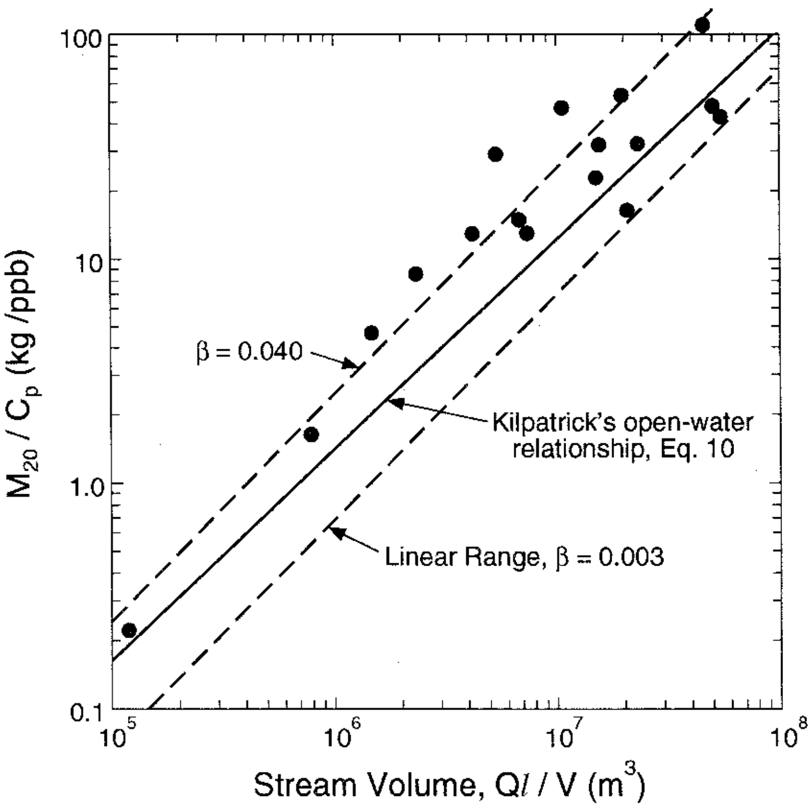
$$M_{20} = 3.8x10^{-6}\ (Q\ell/V)^{0.93}\ C_p\ (open\ water) \qquad (10)$$

in which M_{20}=kilograms of Rhodamine WT dye, in 20% solution, required to produce a peak concentration, C_p (in ppb), at a site located ℓ metres downstream of the point of injection. The discharge Q and velocity V are in m^3/s and in m/s, respectively, so that the quantity $Q\ell/V$ represents the volume of water in the river within the length ℓ, in cubic metres. The above equation was derived from data in the range of stream volumes=$7.5x10^3$-$2.5x10^9$ m^3. Church (1974) presented additional open-water data confirming Eq. 10 for volumes between 10 and $5x10^3$ m^3.

It is noteworthy that the exponent of $Q\ell/V$ in Eq. 10 is close to 1, suggesting the prevalence of linear dispersion conditions. This can be ascertained by examining Eq. 4 which indicates that $M/C_p \propto Q\tau$. In the linear range, $\tau \propto x/V$, hence $M/C_p \propto Q\ell/V$. On the other hand, it could be shown that, in the Fickian range, the exponent would be equal to 0.50.

Data from the ice-covered flow tests are plotted in the form suggested by Eq. 10 in Fig. 8. Though the data points represent the same order of magnitude as what has been found for open-water conditions, Eq. 10 does not provide an adequate quantitative description. The linear-dispersion theoretical relationships that can be obtained from Eq. 4 are also plotted for representative values of ß. Several data points plot above this range, indicating that significant losses occurred in the corresponding tests. Most of these tests were on the Wapiti and Smoky Rivers, under conditions of a very rough ice cover that included frazil ice deposits (Van Der Vinne, 1990).

Fig. 8. Amount of dye needed to produce a specified peak concentration in an ice-covered reach of length *l*. M_{20} = mass of 20% solution of Rhodamine WT fluorescent dye in kilograms.



IMPLICATIONS TO WATER QUALITY MODELLING

The finding that has the most serious implication to modelling longitudinal dispersion in rivers, is the persistence of linear-type dispersion and the complete absence of Fickian dispersion. The latter is, however, assumed commonly in various water quality models, partly because of the simplicity afforded by the implied constant dispersion coefficient. The data obtained so far suggest that this assumption would be unrealistic in most instances.

To be consistent with the results summarized in Eqs. 2-4, the longitudinal dispersion term should be expressed as (see also Beltaos, 1980):

$$<(u-V)(c-C)> = \beta L\{1-\exp(-x/L)\}(\partial C/\partial t) \qquad (11)$$

which in the linear range simplifies to:

$$<(u-V)(c-C)> \approx \beta x(\partial C/\partial t) \qquad (12)$$

Where a single contaminant source is involved, Eqs. 11 and 12 present no particular difficulty because x is singly defined as the distance from the source. However, if two or more sources are considered, a different value of x will apply to each source, whereby it will not be possible to express the resulting concentrations by a single equation of the type of Eq. 1. Instead, a separate equation has to be written and solved for each source, followed by linear superposition of the results in order to arrive at the combined concentration.

To select the value of ß in a particular application, use of Fig. 3 is likely to give the best results. This, however, requires elaborate velocity measurements. An alternative is to use Fig. 7 and select a value in the middle of the range drawn for the winter test data. This will give ß to within a factor of 2 which translates to an uncertainty of ±40% in predictions of concentration and temporal spread.

The available data provide very little guidance with respect to the characteristic length, L. The best that can be done at present is to use Eq. 8, and estimate the coefficient a from the open-water values (0.5-1.8).

SUMMARY AND CONCLUSIONS

Longitudinal dispersion test data in thirteen reaches of ice covered rivers, ranging in width from 40 to 370 m, are discussed. Strikingly non-Fickian behaviour was encountered in these tests but the results were adequately described by the linear dispersion range of a theoretical model developed earlier for open-water conditions.

This model has two parameters, ß and L, that can be evaluated from dispersion test data. However, for tests exhibiting linear dispersion throughout the test reach, only ß can be determined. This parameter quantifies the rate of spread of a contaminant. As in open-water conditions, ß was found to be approximately related to the friction factor of the flow but was somewhat higher than open-water values.

The length characteristic, L, determines the limit of the linear dispersion range while 3L signals the onset of Fickian dispersion. Tests in small, ice-covered streams are needed to ensure that the Fickian dispersion range is sampled so that L can be determined.

One-dimensional modelling of water quality has relied on the conventional dispersion coefficient, D, commonly assumed to be independent of time or distance. The test results indicated that this is rarely the case and therefore the formulation of the dispersion term in the one-dimensional expression of contaminant mass conservation has to be revised. Moreover, where two or more sources are present, a solution has to be obtained for each source separately and the results combined by linear superposition.

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DISCUSSION

Rick Cunjak:

How might frazil ice affect the rate and form of dispersion?

Does your model assume no complexing of pollutant with ice, suspended sediments, etc; how might this affect the dispersion?

Reply:

In many ways - first - because frazil is porous it could act as a temporary storage of pollutant, release it later - secondary peak. Also, frazil can change flow pattern and channels beneath surface ice thereby affecting dispersion.

This is true but the study was meant to look first at physical processing affecting dispersion which will come into play regardless of the chemical properties of a particular pollutant. Precise dispersion must therefore account for different chemical properties of the pollutant.

K.S. Davar:

At the time of collecting field information, was knowledge obtained about the roughness of the undersurface of the ice cover and its relative influence on the dispersion process?

Reply:

No specific effort was made to obtain roughness parameters of the ice cover, beyond a characterization as to its composition, i.e. solid ice, solid plus slush. The composite flow resistance and friction factor was obtained from the slope and average hydraulic radius.

T.D. Prowse:

Could the author please elaborate on why β values for ice-covered situations plot above those for open-water when V_*/V is used in place of the velocity variance.

Reply:

A plausible explanation is as follows. The theoretical model of "linear" dispersion suggests that $\beta \propto$ velocity variance. The latter increases with increasing V_*/V but the relationship is not uniquely defined because additional factors are involved. From a simple analysis of 2-layered, ice-covered flow, it can be shown that, for the same V_*/V, the velocity variance under ice will generally exceed that in open water. Hence, $(\beta)_{ice} > (\beta)_{open}$ for same V_*/V.