DEVELOPMENT OF MATHEMATICAL MODELLING FOR PREDICTION OF RIVER COOLING AND FRAZIL AND ANCHOR ICE FORMATION

by

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ABSTRACT

The physical concept and mathematical relationships for mathematical modelling to predict the cooling of water and the formation of frazil and anchor ice in a river have been developed. The implementation of the above concept and relationships will lead to an operational computerized model to forecast the temperature, the frazil concentration and the thickness and extent of anchor ice along the river as functions of time. The model may be used for real-time management of rivers in winter. During the development of the modelling concept, many important aspects of the physical processes of river cooling, frazil production and distribution, and the formation and growth of anchor ice have been investigated.

INTRODUCTION

For winter river management, it is important to forecast the water temperature along a river at different times and the conditions of frazil and anchor ice in it. In the past, numerous models have been developed to predict the water temperature along a river (see for example, Daily and Harleman, 1972; Paily and Kennedy, 1974 and Giaquinta and Keng, 1978). Models have also been developed to predict, in addition to water temperature, the amount of ice produced in the river (see for example, Poulin et al., 1971; Rumer and Acres, 1973 and Shen, 1980). All the above models were developed for hydraulic design, for environmental impact assessment, for ice boom deployment and removal, and the management of stable river ice covers. The time scale involved in the models, therefore, is large, usually measured in day or days. The time scale of the error, therefore, is also in day or days. As of today, only one highly empirical anchor ice model has been proposed (Marcotte and Robert, 1986). A more refined anchor ice model is surely needed for advancing this scientific frontier.

It is well known in the management of hydro-power stations and municipal water treatment plants that the presence of frazil and anchor ice can severely affect the operation of these plants. Near complete blockage of the water intakes, floods caused by frazil clogging and great flow reductions caused by the combined effect of frazil and anchor ice have long been recorded (see Barnes, 1906, for example). It is also known that the effects of frazil and anchor ice are fast and short term and they usually take place in short reaches of open flows subject to intense chill. Most operational difficulties will diminish as the water temperature increases the next day after sun rise. The management of frazil and anchor ice in rivers therefore requires a short-term, real-time model for short reaches subject to intense cooling. Such a model, however, does not exist.

The mathematical model discussed in this paper was developed to meet the above operational need. The model should lead to short term forecast of the water temperature along a selected river section, as well as the frazil and anchor ice conditions along the river section at different forecast times based on current thermal and meteo-hydrological conditions of the river and the overlaying atmosphere. The forecast would be constantly updated as the environmental conditions change.

Only the physical principles and the mathematical relationships will be shown in this paper. The block diagrams representing the logical connections of these principles and relationships are shown in the report by Tsang (1988a). Computerization of the block diagrams will provide the operational tool for engineers working in river frazil and anchor ice management.

The scenario of the operational room after the implementation and incorporation of the proposed model in river ice management is shown in Fig. 1.

It is seen from Fig. 1 that a meteo-thermal model is assumed to exist by which the heat fluxes to and from the river can be calculated from the real-time meteorological conditions. A hydrological or hydraulic model is also assumed to exist by which the discharge, stage and velocity of the flow at different points along the river section can be found from the hydrological parameters or the operation of the river. The heat fluxes, the discharge, stage, velocity, plus the water temperature recording at one point then become the input to the river cooling, frazil and anchor ice model discussed in this paper. The application of the model to the river from point A to point M will lead to information shown on three TV monitors. The first monitor will show the temperature/distance curves at different forecast times. The present time will be denoted by t_o and the future times will be denoted by $t_1 = t_o + \Delta t$, ..., $t_n = t_n + n\Delta t$, etc. The time increment Δt can be properly selected according to need. The second monitor will show the streamwise distribution of the average concentration of frazil. Again a group of curves for different forecast times will be shown. The final monitor will show the thickness of anchor ice along the river, or the thickness and areal extent of anchor ice in the river section at different times. The constant updating of the three groups of curves according to current hydro-meteorological conditions will permit the river engineer to take proper actions to counteract possible adverse future ice effects.

For simplicity, the model discussed herein will be for one-dimensional flow only. It can be easily expanded to include more natural river flow conditions.

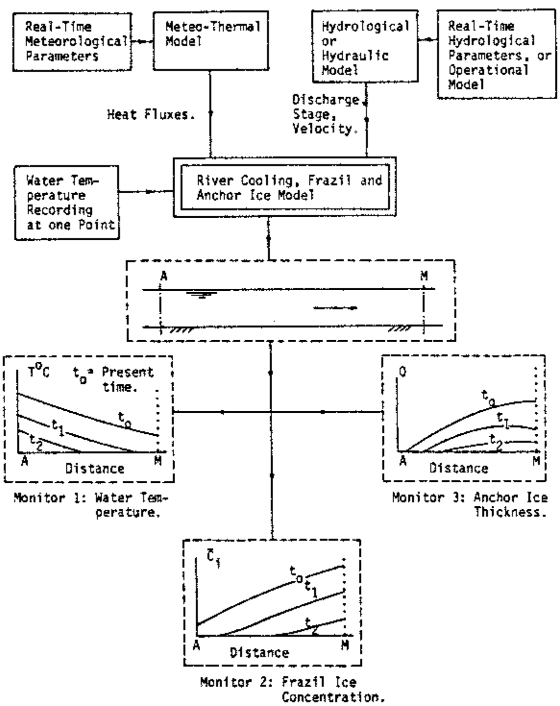

Fig. 1. Scenario in Operation Room after Incorporation of River Cooling and Frazil and Anchor Ice Model.

THEORETICAL BASIS OF MODEL

The cooling of a river and the forming of frazil and anchor ice in it are basically a thermodynamic problem. The loss of heat to the environment is compensated by the sensible heat stored in the water and the latent heat of fusion that is liberated as water is changed into ice.

Fig. 2 shows a control volume of water containing ice. The heat conservation equation for this control volume is given by (Tsang, 1982a)

$$\int_V \rho H_L \frac{\partial C_i}{\partial t} dV + \int_A \rho H_L (C_i \vec{v}) \cdot \hat{n} dA - \int_V \rho c \frac{\partial T}{\partial t} dV - \int_A \rho c (T\vec{v}) \cdot \hat{n} dA - \int_A \vec{q} \cdot \hat{n} da = 0 \quad (1)$$

where

V = the control volume
A = the bounding surface
ρ = the density of water
H_L = the latent heat of fusion of ice
C_i = concentration of ice in water by weight
c = specific heat of water
T = temperature of water
t = time
$\vec{q}$ = Heat flux vector
$\hat{n}$ = unit normal vector pointing away from the bounding surface
$\vec{v}$ = velocity vector relative to the bounding service

In the above equation, the first term is the latent heat of fusion liberated in the volume as ice is formed, the second term is the heat left behind by the departing ice after its formation, the third term is the heat absorbed to increase the temperature of the water, the fourth term is the heat taken away by the departing water from the control volume and the fifth term is the net heat flux flowing away from the control volume across its bounding surface. In the above equation, source and sink terms are not included, but can be added if circumstance warrants.

After converting the surface integrals of the second and fourth terms into volumetric integrations using the Gauss theorem and the application of the equation of continuity to the subsequent equation in the expanded form, from the above equation one obtains:

$$\frac{D}{Dt}\left[\rho H_L \int_V C_i dV\right] - \frac{D}{Dt}\left[\rho c \int_V T\, dV\right] = \int_A \vec{q} \cdot \hat{n}\, dA \quad (2)$$

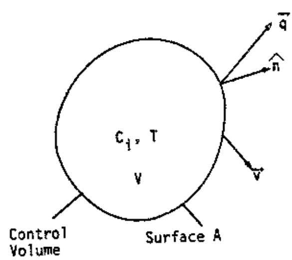

Fig. 2. Thermal Condition of a Control Volume

In obtaining the above equation, the total differentiation sign has been brought out of the integration sign. The above equation can be interpreted as that for a water parcel always containing the same water molecules, the rate of latent heat liberation in it because of ice formation minus the rate of sensible heat absorbed by the water in it is equal to the heat flux leaving the water parcel. It should be noted that in the above equation, the sensible heat of the ice is not differentiated from the sensible heat of the water in the control volume. Such an approximation is acceptable only when the concentration of ice in the water is low. Fortunately, this is usually the case during the formation of frazil and anchor ice in rivers.

When applied to a river section shown in Fig. 3, the above equation may be written in the following form:

$$\rho H_L \frac{D}{Dt} \int_V C_i dV - \rho c \frac{D}{Dt} \int_V T dV = \int_{A_o} q_o dA - \int_{A_i} q_i dA + \int_{A_1} q_1 dA - \int_{A_2} q_2 dA \qquad (3)$$

Where

- A_o = surface area of river section
- A_i = bottom area of river section
- A_1 = cross-sectional area of river section, upstream end
- A_2 = cross-sectional area of river section, downstream end
- q_o = heat flux at river surface
- q_i = heat flux at river bed
- q_1 = heat flux at A_1, and
- q_2 = heat flux at A_2

The heat fluxes are assumed to be perpendicular to the different surfaces.

Because the control volume has been selected to comprise the same water molecules, q_1 and q_2 are necessarily heat fluxes caused by molecular and turbulent heat transfers only and which are proportional to the temperature gradient in the flow direction. Because the temperature gradient of a river in the flow direction usually is quite small and also that q_1 and q_2 tend to compensate each other in their thermal effects to the river section, therefore, unless the river section selected is short, q_1 and q_2 may be neglected in the above equation and one obtains

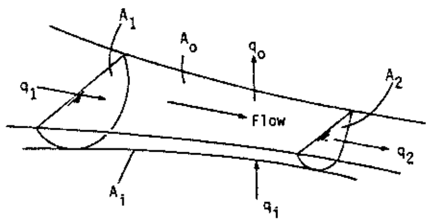

Fig. 3. Cooling Condition of a River Section.

$$\rho H_L \frac{D}{Dt} \int_V C_i dV - \rho c \frac{D}{Dt} \int_V T dV = \int_{A_o} q_o dA - \int_{A_i} q_i dA \qquad (4)$$

Applying the above equation to a one-dimensional flow which is the interest of the present model, if the control volume is chosen to have a length L and an average depth h (See Fig. 4), and the average concentration of ice in it is $\overline{C}_i$ and its average temperature is $\overline{T}$, then the equation will acquire the following form:

$$\rho H_L \frac{D\overline{C}_i}{dt} - \rho c \frac{D\overline{T}}{dt} = \frac{1}{h} (q_o - q_i) \qquad (5)$$

During the mathematical manipulation, the length of the control volume L has been cancelled. The above equation will be the physical basis of the river cooling and frazil and anchor ice formation model discussed herein.

CONCEPTUAL MODELLING OF RIVER COOLING

Theoretical Basis and Discussions

Before the production of ice, Eq. 5 is reduced to

$$\frac{\partial \bar{T}}{\partial t} + u \frac{\partial \bar{T}}{\partial x} = - \frac{1}{\rho c h} (q_o - q_i) \qquad (6)$$

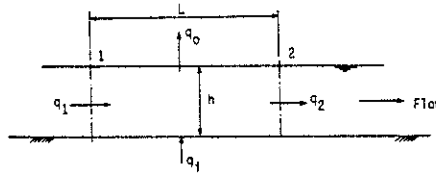

Fig. 4. Cooling of a One-Dimensional Flow.

where x is the direction of flow. This equation provides the theoretical basis for the cooling model to be discussed later.

The above equation, although derived from the Lagrangian point of view, is the same as the equation used by Paily and Kennedy (1974) derived from the Eulerian view point. In Paily and Kennedy's equation, however, the heat flux terms for the two ends of the river section are retained and which leads to a $E\frac{\partial^2 T}{\partial x^2}$ term to the equation, where E is the coefficient of longitudinal turbulent diffusion. In addition, heat source terms are also included in Paily and Kennedy's equation. The inclusion of the above terms in Paily and Kennedy's model is mandatory because their model was developed mainly to predict the thermal effect of waste heat discharged from planned power stations into rivers. Under such conditions, the $\frac{\partial^2 T}{\partial x^2}$ term in the stretch of the river containing the power plant can be large and may not be neglected. For the model consideration here, however, the above situation will not happen because if there is thermal discharge in the river section of interest, then frazil and anchor will be very unlikely to occur and the raison d'être for developing the present model no longer exists.

Paily and Kennedy solved their differential equation numerically with the aid of a computer. Their model requires either the temperatures at the upstream and the downstream ends of the interested river section or the temperature gradients at the two ends as the boundary conditions, plus the initial temperature distribution along the river section as the input. Then it solves the set of differential equations by progressive approximations. Using this model, Giaquinta and Keng (1978) determined the thermal regimes of the Mississippi and Missouri rivers. It should be mentioned that Paily and Kennedy's model is an improved and simplified version of the model developed by Daily and Harleman in that the hydraulic parameters of the river are assumed known, which, in fact is the case for most engineering problems. For Paily and Harleman's model, the hydraulic parameters are obtained by simultaneously solving the open-channel momentum and continuity equations and this makes their model much more complicated.

For operational forecasting, often the initial temperature distribution along the river section is not known (for example, the

Upper Niagara River). The use of the Paily-Kennedy model therefore is difficult. The cooling model proposed here intends to circumvent this requirement and replace it with the temperature history at the upstream end point, which is easier to obtain practically. The mathematical manipulation of the present model will also be simpler than that used in Paily and Kennedy's model in that it does not contain progressive approximation calculations. It must be said that the cooling model proposed here does not intend to replace the Paily-Kennedy model. It is merely a simpler model developed under simpler conditions, namely a short reach of river under intense cooling. The major attention of this paper is on frazil and anchor ice modelling, especially the latter.

River Cooling Modelling

Eq. 6 may be integrated between two streamwise points A and B, with A being the upstream point, and which leads to

$$\bar{T}_{B,t} - \bar{T}_{A,t} = \int_{\Delta x_{AB}} - \{\frac{1}{u_t}[(\frac{\partial \bar{T}}{\partial t})_t + (\frac{q_o - q_i}{\rho c h})_t]\}dx \qquad (7)$$

where the subscript t indicates the time of integration. Where both space and time have to be subscripted, the first subscript will be the space and the second subscript, after a comma, will be the time. Using the averaged values between A and B, the above equation may be written as:

$$T_{B,t} - T_{A,t} = - \frac{1}{\bar{u}_{AB,t}} [(\overline{\frac{\partial T}{\partial t}})_{AB,t} + (\overline{\frac{q_o - q_i}{\rho c h}})_{AB,t}]\Delta x_{AB} \qquad (8)$$

Where the bar indicates averaged values and the subscript AB means the stretch from A to B. For clarity, the bar above the mean temperature T has been dropped.

The above equation is obtained from integrating Eq. 6 for a given instant. If the integration is for a fixed point over a time interval from t_1 to t_2, the following equation will be obtained:

$$T_{x,2} - T_{x,1} = - [(\overline{\frac{\partial T}{\partial x}})_{x,12}\, \bar{u}_{x,12} + (\overline{\frac{q_o - q_i}{\rho c h}})_{x,12}]\, \Delta t_{12} \qquad (9)$$

where the subscript 1 indicates t_1, subscript 2 indicates t_2 and the subscript 12 means the time from t_1 to t_2, or during $\Delta t_{12} = (t_2 - t_1)$.

The river cooling model will be built on the above two equations.

Fig. 5 shows a river section for which the cooling of the water is to be predicted. For this river section, as mentioned in Introduction, a hydrological/hydraulic model is assumed to exist to give the flow velocity and stage at every point. A meteo-thermal model is also assumed to exist to give the heat fluxes to and from the water under various meteorological and hydraulic conditions. The river section is divided by cross-sections A, B, ... N, ..., M into (M-1) sub-sections. Although the flow in each subsection can be a varied flow, it is approximately assumed to be uniform flow.

In Fig. 5, while (a) shows the geometry of the river section, (b) and (c) respectively show the stage and velocity of the flow at instants t_o and t_1. In (d), the heat flux parameter $(q_o - q_i)/\rho ch$, which can be calculated from the hydrological and the meteo-thermal models is shown. In calculating $(q_o - q_i)/\rho ch$, a constant water temperature may be assumed for the whole river section and for all the forecast times. This assumption is permissible for a short river reach under intense chill so that the water temperature variation is small compared to the temperature difference between the air and the water. Finally, (e) shows the temperature of the water along the river section at instants t_o and t_1.

It is mentioned in Introduction that the temperature/time record at one point up to and including instant t_o will also be required as input to the model. In the present modelling, the point chosen is A. The zig-zag arrows in (e) shows the sequence of calculation. From the known temperature at A at time t_o, namely $T_{A,o}$, the water temperature at the same point but the next instant, namely $T_{A,1}$, is calculated. Then, based on $T_{A,o}$ and $T_{A,1}$, the temperature at B but at time t_o, namely $T_{B,o}$, is calculated. Temperature $T_{B,1}$ then is calculated from $T_{A,1}$ and $T_{B,o}$. The calculation continues in the same manner until the temperature at all points and for the two instants are calculated. The calculation of the water temperature for instant t_2 can begin after the water temperatures at t_1 have been calculated. The following sub-sections show the details of the calculation.

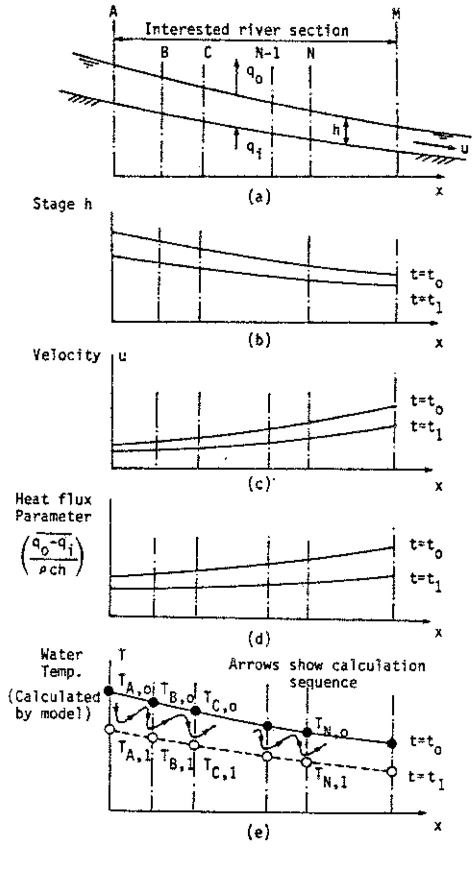

Fig. 5. Working Concept of River Cooling Model

Temperature at A at Future Instants

Fig. 6 shows the time series temperatures recording measured at A upto and including instant t_o at which the water temperature forecast is required. In Fig. 6, the recorded temperature is shown in solid line and the projected temperature beyond t_o is shown by dashed line.

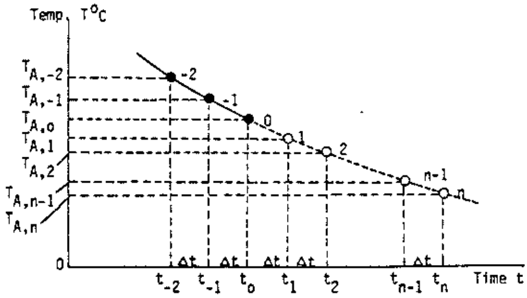

Fig. 6. Projection of Temperature at Cross-section A

If the temperature curve is assumed to be smooth with a continuously varied slope or if

$$\frac{\left(\frac{\partial T}{\partial t}\right)_{A,t+\Delta t}}{\left(\frac{\partial T}{\partial t}\right)_{A,t}} = k;\ k = \text{const} \tag{10}$$

then the following relationship can be obtained:

$$T_{A,n} = T_{A,o} + (T_{A,o} - T_{A,-1})\left\{[e^{-n(k-1)} - e^{-(n+1)(k-1)}]^{-1} - [1 - e^{-(k-1)}]^{-1}\right\} \tag{11}$$

which can be used to calculate the water temperature at any instant t_n.

The above model really amounts to extrapolating the temperature curve. For forecasting times not too far in the future, the temperature curve is likely to follow its functional inertia. As the forecasting time becomes more advanced in the future, greater errors should be expected. The temperature prediction model for point A presented here does not exclude the proposition of other better model, especially those incorporating practical experience gained from the site.

Temperature at Subsequent Points at t_o

From Fig. 5(e), one sees that the temperature at point N can be calculated from the temperatures at the point immediately upstream at instants t_o and t_1. From Eq. 8, for the present situation, one can write

$$T_{N,o} = T_{(N-1),o} - \frac{1}{\bar{u}_{(N-1)N,o}}\left[\left(\overline{\frac{\partial T}{\partial t}}\right)_{(N-1)N,o} + \left(\overline{\frac{q_o - q_i}{\rho c h}}\right)_{(N-1)N,o}\right]\Delta x_{(N-1)N} \tag{12}$$

In the above equation, all the quantities on the right hand side are either known from direct measurement or the hydrological and meteo-thermal models except $\overline{(\partial T/\partial t)}_{(N-1)N,o}$. In the mathematical modelling here, this quantity is approximated by

$$\overline{(\frac{\partial T}{\partial t})}_{(N-1)N,o} = \overline{(\frac{\partial T}{\partial t})}_{(N-1),o1} = \frac{T_{(N-1),1} - T_{(N-1),o}}{\Delta t_{o1}} \qquad (13)$$

which means that the spatial average between (N-1) and N is considered to be equal to the time average between t_o and t_1.

If the distance between (N-1) and N is such that it takes Δt for a parcel of water to travel from (N-1) to N, one should have no difficulty in seeing that

$$\overline{(\frac{\partial T}{\partial t})}_{(N-1)N,o} = \overline{(\frac{\partial T}{\partial t})}_{N,01} \qquad (14)$$

because they both mean the average value of the $(\frac{\partial T}{\partial t})$'s of the same parcels of water. Although for the left hand side, it means all the $\frac{\partial T}{\partial t}$ are taken at time t_o and for the right hand side, the $\frac{\partial T}{\partial t}$ values are taken over a time period Δt, if Δt is not large, the difference should be small. Comparing Eq. 14 with Eq. 13, one sees that the model also approximately equals $\overline{(\frac{\partial T}{\partial t})}_{(N-1),01}$ to $\overline{(\frac{\partial T}{\partial t})}_{N,01}$. If the distance between (N-1) and N is small, the average value of $\frac{\partial T}{\partial t}$ obtained at point N over time period Δt of course should be close to that obtained at point (N-1), otherwise some error should be expected.

The substitution of Eq. 13 into Eq. 12 permits the calculation of $T_{N,0}$. Although theoretically speaking if the time increment and hence the lengths of the subsections are made small, and the thermal extrapolation at A does predict the temperature trend at A, the calculated value of $[T_{N,0} - T_{(N-1),0}]$ should be very close to the measured value obtained from the field. However, for practical engineering reasons, both Δt and the sub-section length will be of reasonably large values. The upstream temperature extrapolation will also bring in error, especially for times far advanced into the future. The above, plus the fact that the hydraulic model and the meteo-thermal model will also generated errors of their own, means that the calculated and measured values of $[T_{N,0}-T_{(N-1),0}]$ will be somehow different. The discrepancy between these two values, however, can be alleviated by introducing a corrective coefficient defined by the following equation:

$$\eta_N = \frac{(T_{N,o} - T_{(N-1),o})_{mea}}{(T_{N,o} - T_{(N-1),o})_{cal}} \qquad (15)$$

where the subscripts "mea" and "cal" mean measured and calculated respectively. The incorporation of the corrective coefficient into Eq. 12 changes it to

$$T_{N,o} = T_{(N-1),o} - \frac{\eta_N}{\bar{u}_{(N-1)N,o}} [(\overline{\frac{\partial T}{\partial t}})_{(N-1),o1} + (\overline{\frac{q_o - q_i}{\rho ch}})_{(N-1)N,o}] \Delta x_{(N-1)N} \quad (16)$$

The value of the η_N's can be obtained from the calibration of the model. It is easy to see from Fig. 5(e) that the range of N for η_N is from B to M, or 2 to M. From the above theoretical derivation, one should also see that the values of the η_N's would be close to unity.

Temperature at Subsequent Points at t_1

From Fig. 5(e), the temperature at point N and at time t_1 can be seen to be given by temperatures $T_{N,o}$ and $T_{(N-1),1}$. According to basic equation Eq. 9, $T_{N,1}$ is given by

$$T_{N,1} = T_{N,o} - [(\overline{\frac{\partial T}{\partial x}})_{N,o1} \bar{u}_{N,o1} + (\overline{\frac{q_o - q_i}{\rho ch}})_{N,o1}] \Delta t_{o1} \quad (17)$$

Again, one sees that all the quantities on the right hand side of the above equation can be measured or calculated except $(\overline{\partial T/\partial x})_{N,o1}$, which may be approximated by

$$(\frac{\partial T}{\partial x})_{N,o1} = (\frac{\partial T}{\partial x})_{(N-1)N,o} = \frac{T_{N,o} - T_{(N-1),o}}{\Delta x_{(N-1)N}} \quad (18)$$

The substitution of Eq. 18 into Eq. 19 and the introduction of another corrective coefficient

$$\psi_N = \frac{(T_{N,1} - T_{N,o})_{mea}}{(T_{N,1} - T_{N,o})_{cal}} \quad (19)$$

into the resultant equation lead to

$$T_{N,1} = T_{N,o} - \psi_N [(\overline{\frac{\partial T}{\partial x}})_{(N-1)N,o} \bar{u}_{N,o1} + (\overline{\frac{q_o - q_i}{\rho ch}})_{N,o1}] \Delta t_{o1} \quad (20)$$

which is the equation for calculating the water temperature at a general cross-section at time t_1. The values of the ψ_N's again, can be obtained from the calibration of the model and will be close to unity according to the theoretical discussions shown above. The range of N, again, is from B to M, or from 2 to M.

After the $T_{N,1}$ values are calculated, or the temperature curve at t_1 is obtained, the $T_{N,2}$ values can be calculated by repeated use of Eq. 20. (Of course, one now substitutes t_1 for t_0 and t_2 for t_1). The same is true for the temperature curves at the subsequent instants. The above means that Eq. 16 is only useful for obtaining the temperature curve at t_0. Should the initial conditions be such that the initial temperature distribution along the river section is known, then Eq. 16 would not be needed and the cooling model shown above would consist of Eq. 20 alone. When compared with the Paily-Kennedy model, one sees that while the Paily-Kennedy model involves computer algorithm, the present model involves only simple and direct calculations.

The proposed model has been compared with the exact analytical solution of Eq. 6 under the simple conditions of $(q_o - q_i)/(\rho q h)$ = const. and an initial constant temperature distribution along the river section. The same temperature distribution curves for future times were obtained for both cases.

The block diagrams for river cooling modelling according to the above theory and for calibrating the model have been constructed by Tsang (1988a). The implementation and computerization of the block diagrams will give the operational model for river cooling forecast.

CONCEPTUAL MODELLING OF RIVER FRAZIL

Physics and Process of Frazil Formation in a River

Before modelling the formation of frazil in a river, the physics and process of frazil formation in a river may be first discussed. For simplicity, it may be assumed at this point that the flow in the channel is uniform, the ambient heat transfer conditions are unchanged and the water temperature at the entrance to the interested section constant. Fig. 7 shows the temperature distribution along such a river from gradually being cooled down to producing frazil.

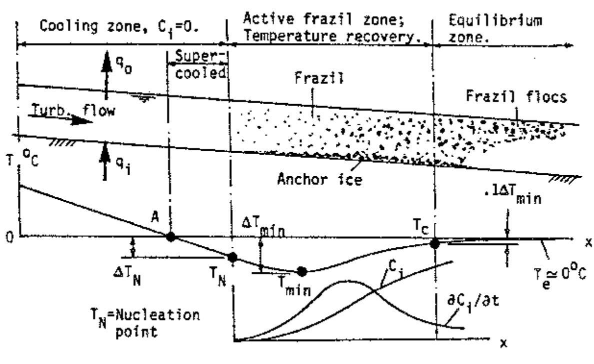

Fig. 7. Cooling of Water and Formation of Ice in a River.

It is seen from Fig. 7 that as water enters the stretch, it is cooled down at a constant rate. At point A, the freezing temperature (nominally 0°C) is reached and the water is capable of producing ice. In general, ice will not be formed until the supercooling ΔT_n is reached at which ice will begin to nucleate on nuclei existed in the water.

As ice begins to form, the latent heat of fusion is liberated. For the short period immediately following the commencement of ice nucleation, because the number of ice crystals in the water is small and hence the rate of latent heat liberation is insufficient to compensate for the rate of heat loss to the environment, the temperature of the water continues to decrease, although at a slower rate. However, as the ice crystals multiply to the point that the rate of latent heat liberation is equal to the rate of heat loss to the environment, the water temperature no longer goes down and the minimum water temperature T_{min} is reached. After the T_{min} point, the water temperature begins to rise because now the rate of latent heat liberation by the numerous ice crystals is greater than the rate of heat loss to the environment. This continues until the equilibrium temperature T_e is reached which is close to 0°C.

Following the formation of frazil, the crystals begin to agglomerate into clusters and flocs upon collision. As the flocs become larger and thicker, they float to the surface to form slushes, which in turn, evolve into pancake ice and frazil floes. Until today, the quantitative relationship governing the evolution of frazil from crystals to floes is still not known, neither is the effect of frazil accumulation and flocculation on the heat transfer between the air and the water. A good understanding of the evolution of frazil, therefore, is necessary before a sound mathematical model can be developed. The frazil modelling discussed here, thus, is subject to the above limitation.

Although frazil crystals float to the surface as soon as they are formed, the turbulence of the flow entrains them back into the lower layers. Therefore, if the evolution state of the frazil does not change and the amount of frazil in the water remains constant, then under a given set of turbulence conditions, as time progresses, an equilibrium state will be reached at which the concentration of frazil will be a function of depth alone.

In reality, however, such a steady state is difficult to obtain because of the continued change of the evolutionary state of the frazil and the continued production of frazil in the river. Only under the conditions of strong turbulent mixing and slow evolutionary changes, can the pseudo equilibrium state be assumed to exist.

Physically, however, it is easy to see that at the beginning of frazil formation, when the frazil is in the form of individual crystals suspended in the water, the frazil crystals will be responsive to all the eddies in the turbulence spectrum, the distribution of frazil in the water column will be more or less uniform. As the frazil crystals agglomerate into clusters and flocs, however, the eddies of sizes smaller than those of the clusters and flocs will no longer be effective in moving the clusters and flocs. More frazil, therefore, will stay near the surface. As the frazil progresses in its evolution, the frazil concentration distribution curve will become more and more skewed towards the surface.

From the above discussions, one sees that to properly model river frazil, two things ought to be looked into, the first being the total amount or the average concentration of frazil at a point and the second being the vertical distribution of frazil at that point.

The distribution of frazil in a turbulent flow had not been studied until quite recently by Tsang (1988b) who studied the vertical distribution of frazil at the equilibrium state. As yet, the study can not be said to be conclusive. In view of such an information gap, the modelling of river frazil is forced to limit to the modelling of the average concentration of frazil alone.

Modelling of River Frazil

To model river frazil, the point of first frazil appearance, or the T_N point has to be determined first. In the modelling presented here, the T_N point is assumed to be the point where the freezing temperature is first reached, or the T = 0°C point. It is seen from Fig. 7 that in reality, the T_N point should be somewhere between the T = 0°C point and the minimum temperature point. For rivers, the maximum supercoolings have seldom been found to be more than 0.1°C. The usual range has been found to be from 0.03°C to 0.06°C. The assumption used here, therefore, can mean the upstream advancement of the first ice appearance point by as much as about 0.05°C. For a river of depth h, flowing at a velocity u and subject to a net heat loss flux of q, to produce a temperature decrease of ΔT, a parcel of water has to travel over a distance L given by

$$L = \frac{\Delta T . u . h}{q} \tag{21}$$

It is known that for a frazil producing river, the net heat loss flux from the river surface is typically in the range of 30 to 50 cal/cm^{2}hr (Tsang, 1982a). With such a heat flux range, for a river of depth h = 5 m and velocity u = 1 m/s, according to the above equation, the water parcel has to travel over a distance in the range of 1,800 to 3,000 m. In other words, in the present modelling, there will be an uncertainty of roughly from 2 to 3 km about the point of first frazil appearance. Although such an uncertainty is substantial, it is acceptable because it gives the most upstream possible point of frazil appearance and thus means conservative modelling. It can be seen from Fig. 7 that the assumption of the T = 0°C point being the first frazil appearance point will not affect the frazil conditions downstream from the actual T_N point.

Tsang and Hanley (1985) have studied the relationship between T_N and T_{min} and found that should T_N = 0°C, the maximum supercooling, or ΔT_{min} would also be close to zero. Based on their study, it is assumed in the present modelling that once frazil begins to appear, the water temperature will remain at 0°C, or in other words, once the water temperature reaches 0°C, it will remain at 0°C.

The relationship governing the change of temperature and the formation of ice in a parcel of water is given by Eq. 5. Prior to the formation of ice in the water, however, the equation is reduced to Eq. 6 which is the basis for river cooling modelling. As said above that following the onset of ice nucleation, the water temperature may be considered to be at the freezing point, the total differentiation of water temperature with respect to time, namely DT/Dt, therefore vanishes from Eq. 5 and the equation is reduced to

$$\rho H_L \left[\frac{\partial \overline{C}_i}{\partial t} + u \frac{\partial \overline{C}_i}{\partial x}\right] = \frac{1}{h} (q_o - q_i) \tag{22}$$

Comparing the above equation with Eq. 6, one sees that the two equations are identical if $H_L\overline{C}_i$ is replaced by -cT or vice versa.

During the development of the river cooling model, although the concentration of frazil is assumed to be zero, there is no constraint placed on the value of the water temperature. The water temperature, therefore, can be either higher or lower than the freezing point although in reality the latter can not happen physically. The range of negative temperature forecast by the model, from the physical stand point, really ought to be the ice-laden zone. According to the discussion shown in the preceding paragraph, one sees that the concentration of frazil in the negative temperature zone can be obtained by simply replacing T with $C_i = -(cT/\bar{H}_L)$.

The assumption that all the river ice is in the form of frazil is permissible for an open channel flow. Field measurements by Tsang (1985) over the Lachine Rapids showed that the average concentration of frazil in the flow could be of the order of 0.1 percent. At a flow velocity of 2 m/s and a flow depth of about 10 m, over a three-day period, should all the frazil accumulate on the river bottom, an anchor ice layer over 5000 m thick would have been formed. Actual field measurements, however, showed that the thickness of the anchor ice was of the order of 1-2 m only. Thus, most of the river ice was in the form of frazil.

From the above discussions, one thus sees that the river cooling model not only predicts the water temperature, but the average concentration of frazil also after slight change in the interpretation of the forecast results or slight modification of the model. There should be no difficulty in seeing that the model is valid both for the forming and melting of ice in the water. Fig. 8 schematicallly illustrates the cooling or warming of water, and the forming or melting of ice in a river.

A block diagram has been developed to expand the river cooling model to include the forecasting of the average frazil concentration in the river also (see Tsang, 1988b). The implementation of the block diagram should give the river frazil forecast model.

A few points are worth discussing at this point to illustrate the inadequacy of the proposed model which is caused by the present poor understanding of the effects of ice presence on heat transfer and on flow behaviour. First, in the present model, the average frazil concentration is obtained by simply assuming that $\bar{C}_i = -cT/H_L$. This means that the heat transfer between an ice infested flow and the environment is simply assumed to be the same as the heat transfer between an ice-free flow and the environment. While for a flow containing only low concentration of frazil, such an assumption is acceptable, for an ice-laden flow of high ice concentration, especially after frazil floes have been formed and they flow sluggishly because of grinding among themselves and with the shores, such an assumption is definitely questionable. Under the latter situation, the heat transfer condition is more like that between a river with an ice cover with the environment. In general, the heat transfer between a frazil-laden flow and the environment varies somewhere between the two extremes of open flow and ice-covered flow. Because the heat transfer between an ice laden flow and the environment is not well understood, a better modelling of the average concentration of frazil in the flow can not be developed at this point.

The presence of ice in the flow should also affect the flow characteristics. For the Beauharnois Canal, Tsang (1982b) found that the production of frazil in the canal could increase the resistance of the canal by as much as 40 percent. For the Niagara River, Ontario Hydro (1969) has also measured a flow reduction of 25 percent because of frazil and anchor ice formation. Thus, under the frazil-producing conditions, both the flow velocity and stage will be different from that of an ice-free flow. The ice effects, however, have not been incorporated into river cooling modelling. Their subsequent effects on river frazil prediction, therefore, have also been overlooked. The accuracy of the prediction of frazil in the river forecast by the proposed model, thus suffers from the above shortcomings.

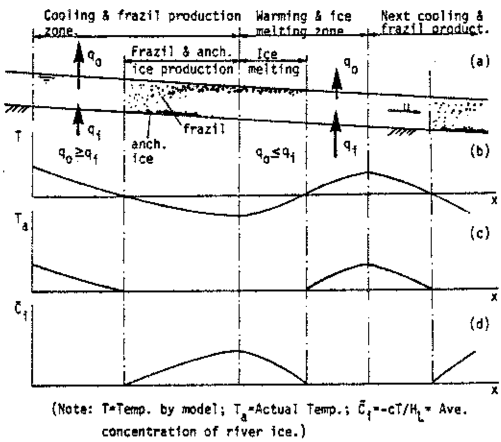

Fig. 8. Illustration of prediction of cooling and warming of water, and the forming and melting of ice.

Again, the above weakness of the model may be alleviated somewhat by calibrating the model under ice conditions. The block diagram for calibrating the model was given by Tsang (1988a). Although the calibration of the model appear to be simple and straightforward in theory, its implementation can be difficult because of the demanding logistic requirements. The corrective coefficients obtained also may not be used under different ice conditions. It is best to put more effort in studying the basics of a frazil-laden flow.

CONCEPTUAL MODELLING OF ANCHOR ICE

The Physics and Process of Anchor Ice Formation in a River

Fig. 7 also shows the formation of anchor ice in a river. It shows that as the water is cooled down to the nucleation point, frazil will begin to form and flocculate. Immediately following the T_N point, the water temperature will continue to go down until the maximum supercooling point is reached. After that, the water temperature will rise again until the equilibrium temperature is reached. Should the atmospheric conditions be such that heat flows from the water to the air, the equilibrium temperature would be sub-zero and ice would continue to form in the water. On the other hand, should the heat flow from the air to the water, then the equilibrium temperature would be above zero and ice would be melting in the system.

Frazil is only active in a supercooled environment in which the frazil crystals will adhere to each other and to objects of other materials. When the crystals or flocs adhere to the river bottom, anchor ice is formed.

Immediately following its initial nucleation, frazil is in the form of fine individual crystals suspended in the water and the total amount of frazil in the water is small. Because the motion of the small frazil crystals is affected by all the eddies in the turbulence spectrum of the flow and little energy is needed to pull the small amount of buoyant crystals from the upper to the lower layers, the distribution of frazil in the water column is fairly uniform at this time. As time progresses, more frazil will be produced in the water and this leads to a higher concentration of frazil in the water as a whole. On the other hand, the frazil crystals will also have more time to agglomerate into flocs and the motion of the flocs is only affected by the eddies in the turbulence that are comparable in size or larger than the size of the flocs. In other words, the turbulence energy associated with the small eddies of the turbulence is no longer effective in redistributing the frazil in the water column, but gets damped out in the frazil flocs. The above thus leads to a progressively lower concentration of frazil in the lower water layers and relatively high concentration of frazil in the upper layers. As far as the concentration of frazil at the bottom is concerned, in the early period immediately following the initiation of frazil nucleation, the increasing effect on frazil concentration due to higher frazil production outweighes the reducing effect because of agglomeration, the bottom concentration of frazil, will increase with time, until the point of maximum bottom frazil concentration is reached. Thereafter, however, the negative effect of reduced turbulence action will overtake the positive effect of increasing frazil loading and the bottom concentration of frazil will begin to reduce. Sufficiently downstream from the point of initial frazil nucleation, the bottom concentration will be reduced to the point of practically zero. The variation of the vertical frazil concentration profile in the direction of the flow is schematically shown in Fig. 9.

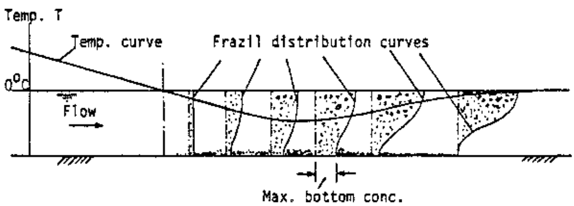

Fig. 9. Streamwise Variation of Vertical Distribution of Frazil and Anchor Ice.

At a point on the river bottom, anchor ice can grow by two mechanisms, the adhesion and entrapment of frazil to the river bottom and/or to the existing anchor ice, and the growth of the ice crystals forming the anchor ice in the supercooled flow. Field investigations by Ontario Hydro (1967, 1968, 1969, 1970), however, showed that the second mode of growth is of secondary importance only. The growth of anchor ice, therefore, can be approximated by the adhesion and entrapment of frazil to the river bottom or the existing anchor ice alone. Let the concentration of frazil at the river bottom be C_{ib}, the turbulence velocity (rms turbulence velocity) at the bottom be w' and the porosity of the existing anchor ice be p, then the rate of increase of the anchor ice thickness θ should be given by the following equation:

$$\frac{\partial\theta}{\partial t} = \frac{\beta}{1 - p} \cdot C_{ib} \cdot w' \quad (23)$$

where β is a coefficient describing the ability of the frazil crystals to adhere to and being trapped by the river bottom or the existing anchor ice. The above equation simply states that the rate of increase in thickness of the anchor ice is equal to the flux of frazil reaching the bottom after the effects of porosity and adhesion and entrapment have been taken into account. After absorbing p into β, Tsang (1988a) showed that the above equation can be changed to the following form:

$$\frac{\partial\theta}{\partial t} = \beta(\Delta T, u_b^2, R, t_*) \cdot C_{ib} \cdot u_* \quad (24)$$

where ΔT is the supercooling of the ambient flow, u_b is the relative velocity between the flow and the anchor ice, R is the flow Reynolds number, t_* is the time lapsed from the time of frazil nucleation to the time when the frazil crystals reaching the interested point and u_* is the friction velocity. The integration of the above equation with respect to time, i.e.,

$$\theta(x,t) = \int_0^t \beta(\Delta T, u_b^2, R, t_*) \cdot C_{ib} \cdot u_* \, dt \quad (25)$$

thus will give the thickness of the anchor ice as a function of time and position.

It may be mentioned that Marcotte and Robert (1986) have tried to model the formation of anchor ice. In their model, anchor ice is assumed to grow by the growth of the ice crystals that form the anchor ice alone. In estimating the heat loss to the flow that is provided for by the latent heat of fusion released by the forming anchor ice, they considered it is caused by the temperature difference between the flow and the river bottom. For the former, they considered it is equal to that as calculated by heat flux balance (or the negative temperature shown by the rivering cooling model proposed here, please see Fig. 8b) and for the latter, they assumed it is equal to 0°C.

Limitation on Thickness of Anchor Ice

Once formed, anchor ice will be subject to a buoyant force because it is lighter than water. In addition, the drag of the flow also tends to tear it away from its anchoring place. The bond of the anchor ice to the river bottom resists the combined action of the above two forces. For simplicity, the flow drag is overlooked in the present modelling. It can be easily incorporated into the model when the need arises.

To theoretically study the lifting of anchor ice by buoyant force, one may approximately assume that the river bed is covered with a layer of round sediment particles of radius r and which are well packed in the manner as shown in Fig. 10. The sediment particles are also assumed to be non-cohesive. From the diagram one sees that the effective area occupied by one sediment particle is $2\sqrt{3}\ r^2$.

The buoyant force acted on each sediment particle therefore is

$$2\sqrt{3}\,\theta\, r^2 g\,(\rho_w - \rho_i)(1 - p)$$

and this buoyant force is counter-balanced by the weight of the sediment particle in water

$$\frac{4}{3}\pi r^3 g(\rho_s - \rho_w)$$

where ρ_s is the density of the sediment. From the balance of these two forces, one obtains

$$\theta = \frac{2\pi}{3\sqrt{3}}\left[\frac{\rho_s - \rho_i}{\rho_w - \rho_i}\right]\frac{r}{1 - p} \qquad (26)$$

With a typical density of sediment of $\rho_s = 2.63$ g/cm^3 and a porosity of 0.5, from the above equation, one can calculate the thickness of the anchor ice for lifting up the sediment to be $\theta \simeq 50$ r. In other words, the preliminary calculation here shows that the river bottom sediment would permit the accumulation of anchor ice roughly to 25 times their diameter. For 1 cm gravel, say, the maximum anchor ice thickness would be about 25 cm. For 10 cm rocks, the maximum anchor ice would be 2.5 m, etc.

Field observation often showed that anchor ice can float up sediment particles or rocks several times the size given by the above criterion. This could happen if the river bottom was not composed of sediment particles of the same size and the particles are not packed as tightly as shown in Fig. 10.

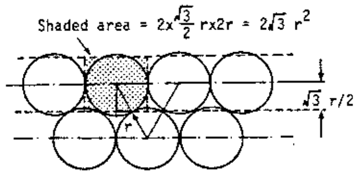

Fig. 10. Packing of Sediment Particles on River Bottom.

Because rivers of heavy sediment bottom permit the build up of thick anchor ice, ice islands grown from anchor ice in shallow and fast reaches where the bottom is often lined with large stones therefore are to be expected, and are indeed often observed in the nature. In this case, the flow depth provides the upper limit to the thickness of the anchor ice.

The peeling off of anchor ice from the bottom provides the second limiting factor on the thickness of the anchor ice. For rivers with rocky bottoms, the peeling off is the only controlling factor of anchor ice thickness. For rivers with sediment bottoms, the peeling off can occur when the bonding force between the anchor ice and the sediment particles is less than the weight (immersed weight) of the particles. As of today, little has been done to study the bonding of anchor ice to the river bottom under different parametric conditions. However, it has been well observed and documented (see Ontario Hydro, 1967, 1968, 1969, 1970, for example) that following sunrise and the warming up of the water, the river bottom will release the anchor ice at a progressively faster rate as the day advances. The release of anchor ice stops quickly as the water temperature decreases to the freezing point again as the evening approaches. This observation leads one into thinking that the bonding strength is a function of the positive heat gained by the anchor ice during the period when the water temperature is higher than 0°C.

Let the water temperature be T and the temperature of the anchor ice be 0°C, the heat flowed from the water to the anchor ice per unit surface area over a time period t will be given by $K\int_0^t Tdt$, where K is the heat transfer coefficient. Assuming that this heat is equally shared by all parts of the anchor ice, then each layer of unit thickners, including the bottom layer clinging to the river bottom, will receive a heat gain of $K\int_0^t Tdt/\theta$. Based on Ontario Hydro's observations, one may assume as a first approximation the simplest functional relationship, namely a linear relationship, between the bonding strength f_b and $\int_0^t Tdt/\theta$. This linear relationship can be written as

$$f_b = \frac{f_b^*}{Q^*} (Q^* - \int_0^t Tdt/\theta) \qquad \text{for } T > 0 \qquad (27)$$

and graphically shown in Fig. 11. It is seen from Fig. 11 that in a supercooled flow, the bonding strength is equal to f_b^*. When the accumulated heat to the anchor ice per unit area and per unit thickness reaches Q^*, the bonding strength of the anchor ice is reduced to zero.

When the buoyant force of the anchor ice $g(\rho_w - \rho_i)\theta$ is greater than the bonding force, or when

$$\theta > \frac{f_b^*}{g(\rho_w - \rho_i)Q^*} [Q^* - \int_0^t Tdt/\theta] \qquad (28)$$

the anchor ice will be peeled off the river bottom. Some sort of experiments or field observations will have to be made to establish the values of f_b^* and Q^* before the model proposed here can actually be implemented.

Before concluding this section, it may be of interest to mention that Marcotte and Rovert (1986), based on

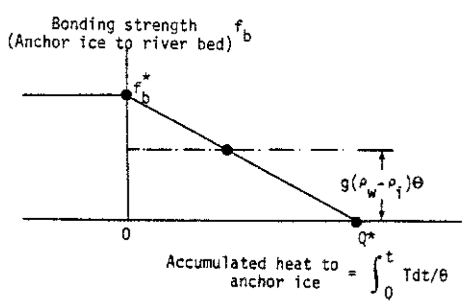

Fig. 11. First Approximation Model for Bonding Strength of Anchor Ice

observations from the Lachine Rapids, assumed that the fraction of the anchor ice released by the warm water in one hour is equal to twice the average temperature during the preceding hour.

Anchor Ice Modelling

Because of the present knowledge gap, one neither knows the distribution of frazil in a flow under different parametric conditions from which C_{ib} may be determined, nor the characteristics of frazil adhesion and entrapment to anchor ice by which β may be estimated, the integration of Eq. 25 to obtain the thickness and areal distribution of anchor ice in a river section is impossible. Facing with this difficulty, the following approximate model is proposed.

It is seen from Figs. 7 and 9 that following the nucleation of frazil there is a zone in which frazil will continue to form and adhere to the river bottom to form anchor ice. Because in this zone the frazil crystals are active in that they adhere to each other and to the river bottom easily, the surrounding water is necessarily supercooled. In studying the formation of frazil in a flow under differfent supercooling conditions in the laboratory, Tsang and Hanley (1985) found that there is a characteristic time t_c after the initial nucleation of frazil at which the water temperature will approximately return to 0.1 the maximum temperature depression, namely at 0.1 T_{min}. The zone covered by t_c may be approximately considered to be the active frazil zone. An empirical equation has been established relating t_c to T_N. For T_N = 0°C, t_c was found to be about 105 seconds, or approximately 2 minutes. It should be mentioned that in Tsang and Hanley's experiments, the rate of cooling of the water was about an order of magnitude greater than that which normally happens in the field. Some room of future refinement, therefore, exists in this regard. Based on the above facts, the anchor ice conceptual model as shown in Fig. 12 is proposed.

The conceptual model assumes that frazil begins to nucleate as soon as the water temperature reaches the freezing point, or that T_N = 0°C. Following the initial nucleation point is a zone covered by the characteristic time t_c of about 2 minutes, which means that it will take about 2 minutes for water to flow through this zone. It is assumed that only within this zone will the frazil be capable of adhering to and being trapped by the river bottom or the anchor ice that had already been formed on the river bottom. Beyond it the frazil is assumed to have agglomerated into large flocs and floated to the surface to form frazil slushes. The presence of frazil at the bottom, therefore, will no longer be felt and anchor ice will not be produced. It is further assumed that within this zone, the distribution of frazil is uniform and equal to the average concentration of frazil at time $t_c/2$ after the initiation of frazil nucleation. As the weather changes, the temperature curve will fluctuate and this leads to the movement of the T_N point, and hence the uniform frazil suspension or the active frazil zone. The moving back and forth of the active frazil zone is like the movement of a paint brush painting anchor ice on the river bottom. Where the brush touches, there will be anchor ice. In places where the brush paints over more often, the anchor ice will be thicker. The thickness of the anchor ice continues to increase until it is overcome by the buoyant force as depicted by Eqs. 26 and 28.

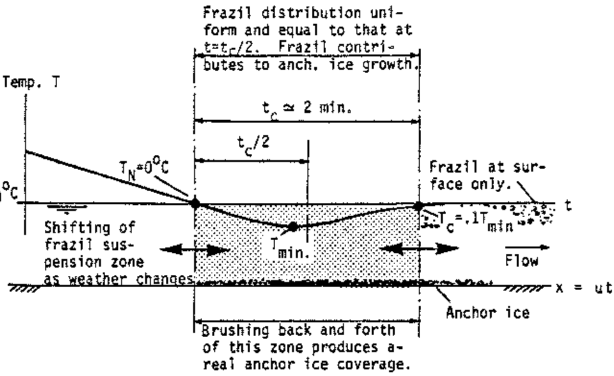

Fig. 12. Illustrative Diagram for Anchor Ice Modelling.

Denoting the average concentration of frazil at $t_c/2$ after the $T_N = 0°C$ point by $(\bar{C}_i)_{t_c/2}$, from Eq. 23, one sees now that the increase of the anchor ice thickness at a point is given by

$$\Delta\theta = \beta_N \cdot (\bar{C}_i)_{t_c/2} \cdot u_* \cdot \Delta t \qquad (29)$$

where β_N is the adhesion and entrapment coefficient for the Nth sub-section and which should be site specific and calibratable.

Because of the frazil model mentioned earlier will forecast the points where the freezing temperature is first reached, as well as the average concentration of frazil at any instant, the incorporation of the concepts shown above for anchor ice modelling, therefore, should not be too difficult and indeed has been done by Tsang (1988a). The implementation of the block diagram constructed by Tsang in the above mentioned report and the calibration of the model should give a working model for anchor ice forecasting.

CONCLUSIONS AND DISCUSSION

The preceding sections showed that it is possible to model the short-term phenomena of river cooling, frazil production and anchor ice formation. The ground work for formulating the models has been laid and the implementation of which will lead to working models for real-time river cooling and frazil and anchor ice formation forecasting. Because the present meagre knowledge on many aspects of frazil and anchor ice, some simplifications and approximations had to be made to arrive at implementable models. As new scientific knowledge is discovered and incorporated into the mathematical concepts, the shortcomings of the present model can be gradually improved. The present paper thus not only serves to lead to a working model for short-term forecasting of river cooling and frazil and anchor ice formation, it also serves to identify the missing links of the science on frazil and anchor ice that future efforts should be directed.

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